Weak Error Analysis and Small-Noise Optimization of Stochastic Linear Multistep Methods (SLMMs) for Itô and Lévy SDEs

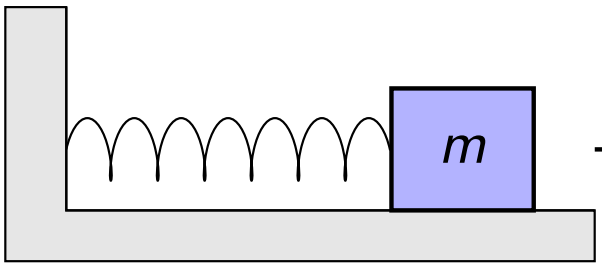
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Deterministic Processes

We know how to model deterministic processes.



$$\xrightarrow{F = ma} \frac{d^2x(t)}{dt^2} = -\frac{k}{m}x(t)$$

Brownian Motion (Wiener Process)

Real-world processes have randomness.

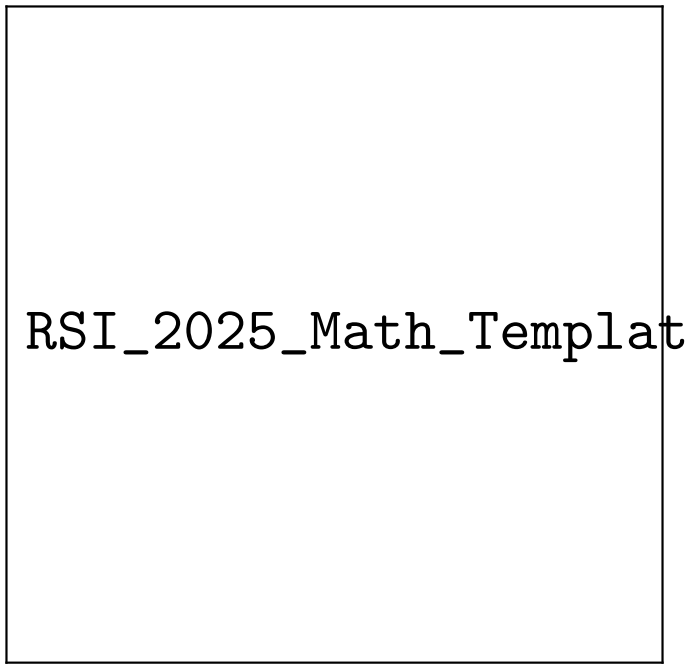


Figure 1: Particle Interaction System

Stochastic Differential Equations

Ordinary Differential Equation (ODE):

$$\frac{dx(t)}{dt} = f(x(t), t) \longleftrightarrow dx(t) = f(x(t), t)dt.$$

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Stochastic Correction:

$$dX_t = f(X_t, t)dt + g(X_t, t)dW_t.$$

We use X_t instead of $x(t)$ to denote a *stochastic process*.

Example (Black-Scholes Equation)

For a stock price X_t,

$$dX_t = \mu X_t dt + \sigma X_t dW_t$$

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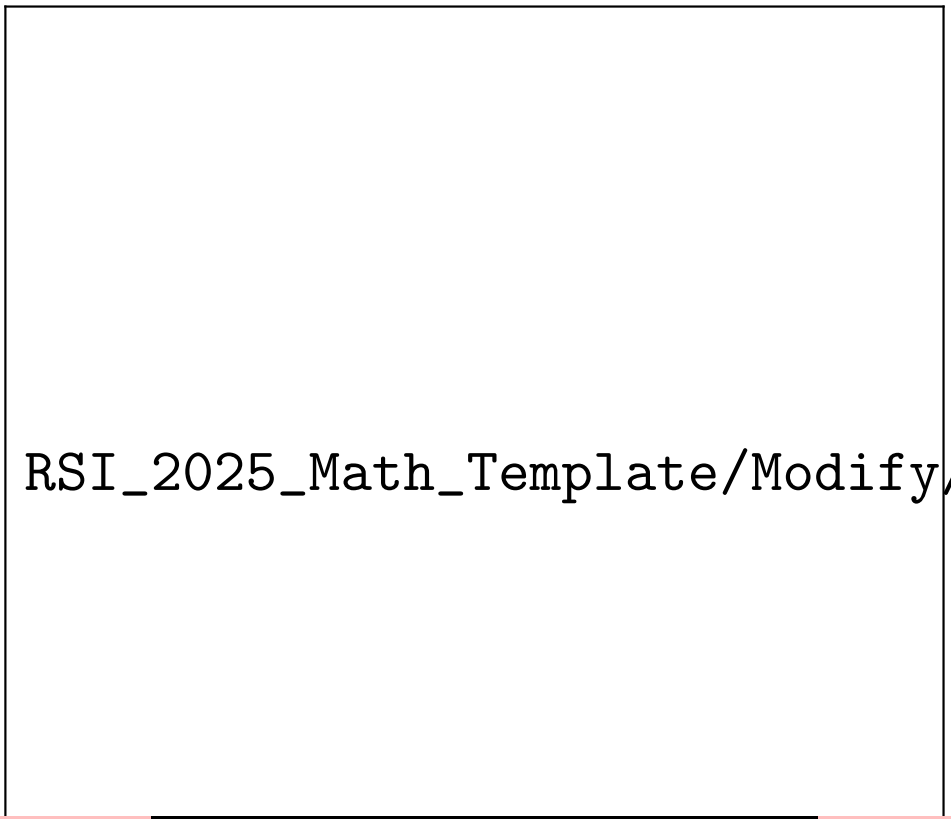
$$\implies X_t = X_0 \exp((\mu - 1/2\sigma^2)t + \sigma W_t)$$

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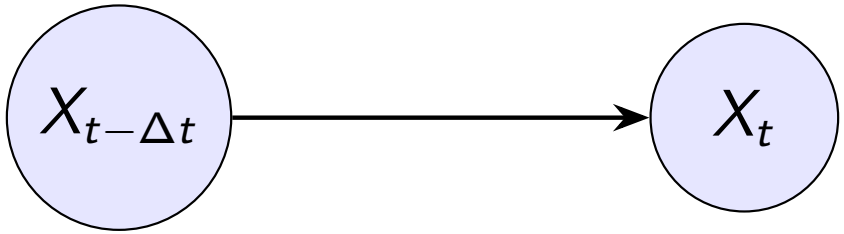


The Problem

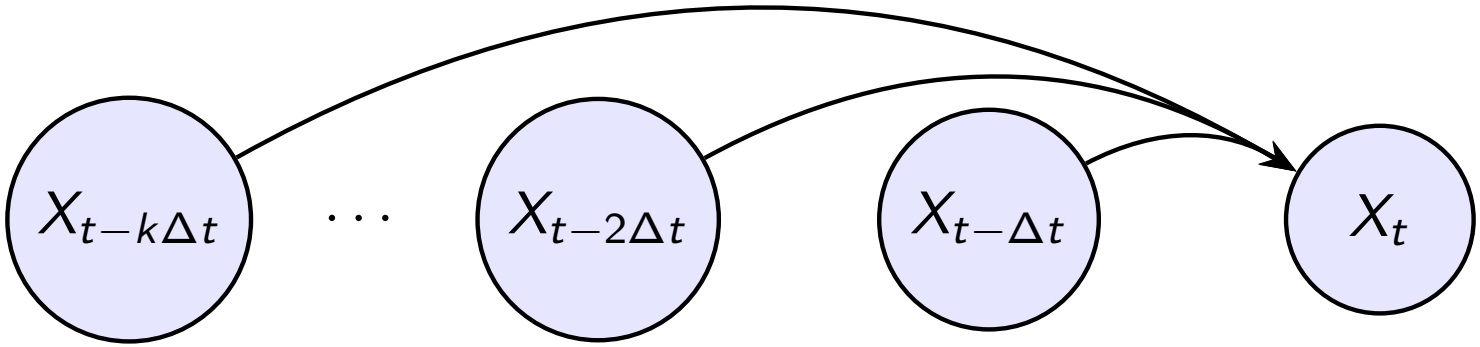
Goal: Approximate solution X_t with process $\overline{X}_t$ "close" in distribution at a low computational cost.



One-Step Methods



Multi-Step Methods



Multi-Step Methods

One-Step Euler-Maruyama:

$$\overline{X}_{t+\Delta t} = \overline{X}_t + f(\overline{X}_t, t)\Delta t + g(\overline{X}_t, t)\Delta W_t.$$

Multi-Step Methods

One-Step Euler-Maruyama:

$$\overline{X}_{t+\Delta t} = \overline{X}_t + f(\overline{X}_t, t)\Delta t + g(\overline{X}_t, t)\Delta W_t.$$

Two-Step Generalization:

$$\overline{X}_{t+\Delta t} = \overline{X}_t + \Delta t \left(\frac{3}{2} f(\overline{X}_t, t) - \frac{1}{2} f(\overline{X}_{t-\Delta t}, t - \Delta t) \right) + g(\overline{X}_t, t)\Delta W_t.$$

A "Closeness" Metric

Weak error expansion:

$$\mathbb{E}[\varphi(X_T)] - \mathbb{E}[\varphi(\overline{X}_T)] \sim C_{\varphi,\rho}\Delta t^{\rho} + C_{\varphi,\rho+1}\Delta t^{\rho+1} + \cdots$$

with smooth φ.

- Constant Factor Improvement: Decrease $C_{\varphi,\rho}$
- Asymptotic Improvement: Increase ρ

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- Is there improvement over one-step methods?
- **Should we use multistep methods in practice?**

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Theorem (A., Lee-2025)

There is a surefire method to compute an arbitrarily high accuracy weak error expansion of any linear multistep method.

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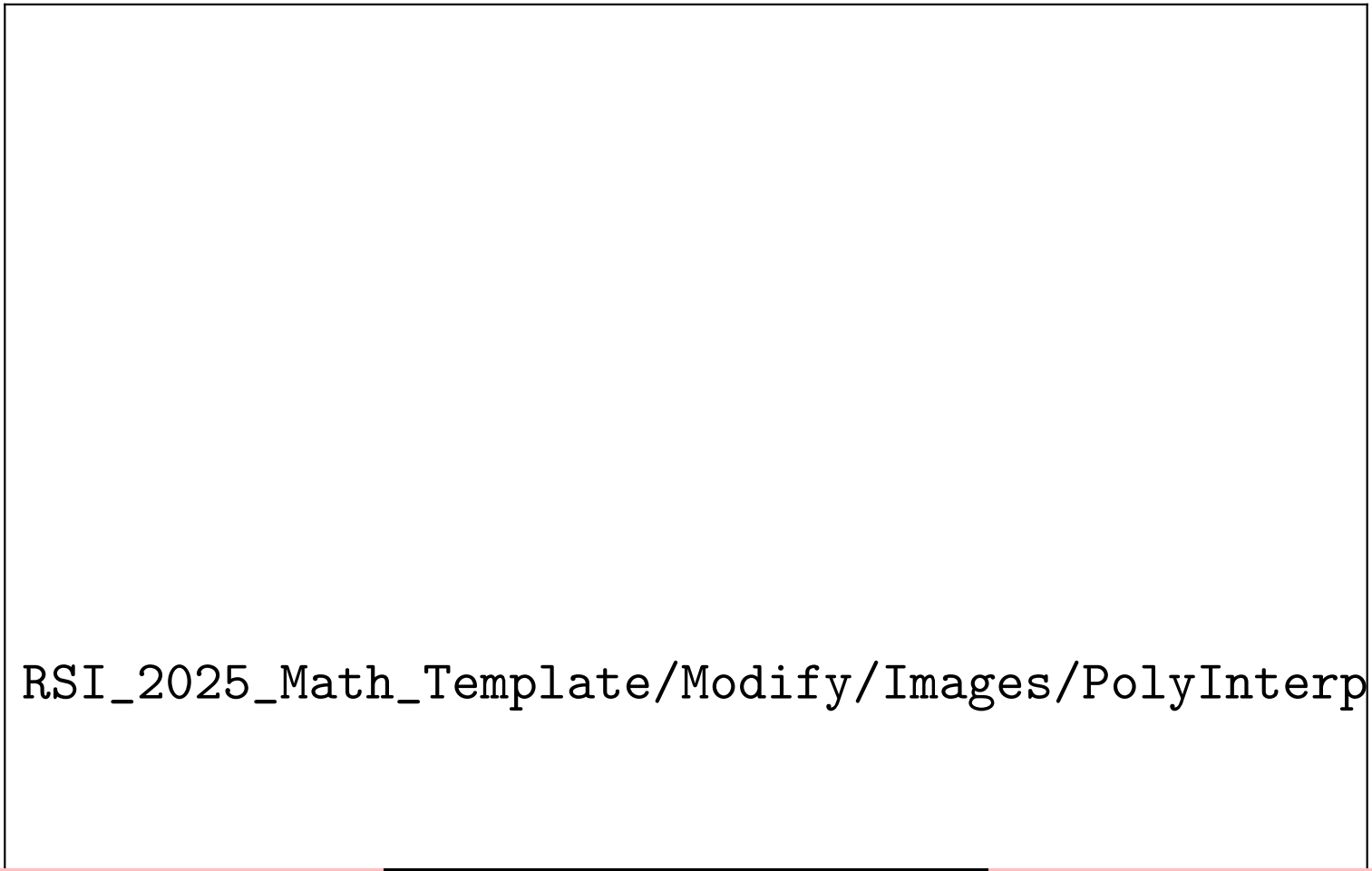
$\Longrightarrow$ **Constant Factor Improvement**

Key Idea: Correspondence between Weak Error of SDEs and Elliptic PDEs given by Feynman-Kac Theorem.

Results

Theorem (A., Lee-2025)

When the noise is "small" one can construct multistep methods that make **asymptotic improvement** in error over their one-step analogues.



Results

We have analogous results for Lévy-driven SDEs.

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Results

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For classical *and Lévy-driven* SDEs,

- we can compute the weak error of any linear multistep method,
- linear multistep methods approximate solutions better than their one-step analogues, especially in the small-noise case,
- results hold true empirically, so...

We should use multistep methods in practice.

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